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STABILITY OF DIFFERENCE EQUATIONS
SELECTED TOPICS

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STABILITY OF DIFFERENCE EQUATIONS
SELECTED TOPICS

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ABSTRACT

In Part I, an elementary review is given concerning the stability properties of some simple linear partial difference equations. Part II contains examples illustrating some properties of typical nonlinear partial difference equations and an analysis for predicting these properties.

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INTRODUCTION

In many cases, the changes with time of a physical situation can be represented well by a set of partial differential equations, but often the appropriate equations are much more easily derived than solved. Analytical methods for obtaining solutions can become impossibly complicated for various reasons. The equations may, for example, be nonlinear, or even if linear, they may be subjected to initial conditions of great complexity. Thus it is sometimes necessary to use numerical methods.

In a commonly-used class of numerical methods, the differential equations are replaced by finite difference approximations, and the solution is obtained by algebraic processes, stepwise through time. (Detailed illustrations are given in Part I.) The resulting solution is not exactly that of the differential equations, but may be close provided that proper care is taken. It is necessary that the increments of independent variables be small compared to the structure of solution which must be resolved. But such a requirement is not sufficient to guarantee accuracy; in many cases the difference representation can be unstable, so that any perturbation will be amplified indefinitely, obscuring all realistic features.

The stability properties of linear partial difference equations can be explored by well-known techniques. Considerable discussion and many references have been given by Richtmyer,[1] and this report includes in Part I some of the techniques as applied to several representative examples. Concerning the effects of nonlinearity on stability, relatively little has been written. Part II is concerned with some of these effects.

PART I

STABILITY AND ACCURACY OF LINEAR EQUATIONS

To illustrate the methods of solution by finite-difference approximation, and to exhibit some of the properties of the techniques, we consider several simple examples. The results will be useful for comparisons with those of Part II.

Example 1. The one-dimensional partial differential equation for the diffusion of temperature in a material with constant positive coefficient of diffusion, a, is

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2} \tag{1}$$

where T is the temperature and x and t are the independent space and time variables. This equation is linear in the unknown temperature function, so that the sum of several solutions is also a solution.

The differential equation is stable in the following sense: If a certain initial-condition boundary-value problem is posed, and the resulting solution is compared with that obtained from slightly different initial conditions, the two solutions will be only slightly different. We shall demonstrate this fact for comparison with later results in which such stability is not present.

The difference between the two solutions of (1), arising from the two different initial conditions, can be written

$$\delta T = \sum A_{k,\omega}\, e^{ikx}\, e^{\omega t}$$

which is simply a Fourier decomposition of δT, in which the sum is over an appropriate set of values of k and ω. The difference itself must satisfy (1), and so indeed must each term in the Fourier sum. Substitution into (1) produces the result that each Fourier term is a solution provided that

$$\omega = - a k^2 \tag{2}$$

This relation shows that for any value of k, $\omega \leq 0$ so that the term decays exponentially to zero as time increases; or in the case $k = 0$ it remains constant, corresponding to a constant difference everywhere between the two solutions. Thus the two solutions, subsequent to initial time, approach each other to within a constant difference which can be no larger than the average of the initial difference.

Consider now a finite-difference method for solving (1) approximately. To be specific, suppose that the material extends from $x = x_L$ to $x = x_R$ and that boundary conditions are supplied for those two ends, and initial conditions for the intermediate part. To visualize the finite difference procedure, imagine the material to be divided into J equal space intervals of length $\delta x = (x_R - x_L)/J$. The resulting cells are labeled with the index $j = 1,2,\cdots,J$. Define the "temperature for cell j" by the relation

$$T_j \delta x \equiv \int_{(x_j - \frac{\delta x}{2})}^{(x_j + \frac{\delta x}{2})} T \, dx$$

where x_j is the distance to the center of cell j. Then, from (1),

$$\frac{dT_j}{dt} = \frac{a}{\delta x}\left\{\left[\frac{\partial T}{\partial x}\right]_{x=x_j+\frac{\delta x}{2}} - \left[\frac{\partial T}{\partial x}\right]_{x=x_j-\frac{\delta x}{2}}\right\}$$

This equation is exact. The next step is to introduce the approximation

$$\left[\frac{\partial T}{\partial x}\right]_{x=x_j\pm\frac{\delta x}{2}} = \pm\frac{1}{\delta x}\left[T_{j\pm 1} - T_j\right]$$

so that

$$\frac{dT_j}{dt} = \frac{a}{\delta x^2}\left[T_{j+1} + T_{j-1} - 2\,T_j\right] \tag{3}$$

The final step in the procedure is to divide the time after initial time ($t = 0$) into finite intervals of constant duration δt each. We count the time cycles by index n, so that $T_j^n \equiv T_j$ at $t = n\,\delta t$. Then the time derivative can be approximated by $\left(T_j^{n+1} - T_j^n\right)/\delta t$, and the finite difference equation becomes

$$T_j^{n+1} = T_j^n + \frac{a\,\delta t}{\delta x^2}\left[T_{j+1}{}^n + T_{j-1}{}^n - 2\,T_j^n\right] \tag{4}$$

The above procedure is arbitrary in several respects; various alternative difference approximations could have been obtained, for example

$$T_j^{n+1} = \frac{1}{2}\left(T_{j-1}{}^n + T_{j+1}{}^n\right) + \frac{a\,\delta t}{\delta x^2}\left[T_{j+1}{}^n + T_{j-1}{}^n - 2\,T_j^n\right] \tag{5}$$

or

$$T_j^{n+1} = T_j^n + \frac{a\ \delta t}{\delta x^2}\left[T_{j+1}^{n+1} + T_{j-1}^{n+1} - 2\ T_j^{n+1}\right] \tag{6}$$

or an infinite number of others. They must all satisfy the requirement of formal reduction to (1) as δx and δt become vanishingly small.

Equations (4) and (5) are called "explicit." In both cases the temperature for each cell for the new time cycle can be found from information from the old cycle simply by algebraic substitution. Equation (6), on the other hand, is in "implicit" form. Solution for the new temperatures requires somewhat more complicated techniques which are straightforward, but which are not discussed here. We shall see that in some cases the implicit form offers advantages of stability which may overcome the disadvantages of solution complexity.

The finite difference equations as written above force consideration of two additional fictitious cells, numbers $j = 0$ and $j = J + 1$. The temperatures of these cells are needed for computing the new temperatures for cells $j = 1$ and $j = J$. The difficulty is resolved by a proper adaptation of the desired boundary conditions. Suppose, for example, that a variable temperature $T_L(t)$ is specified for the point $x = x_L$. Then T_o^n may be determined by requiring $T_L(n\delta t)$ to be the average of T_o^n and T_1^n

$$T_o^n \equiv 2\ T_L(n\delta t) - T_1^n$$

If the temperature gradient $G_L(t)$ is specified at that boundary (the gradient is directly proportional to the heat flux), then T_o^n

is easily found from the relation

$$\frac{T_1^n - T_o^n}{\delta x} = G_L(n\delta t)$$

Next we consider the stability properties of some of the difference equations. As in the differential stability analysis, the difference between two solutions can be decomposed into Fourier components, of which a typical one is

$$Ae^{ikj\delta x}\, e^{\omega n\delta t}$$

Substitution of this into (4) leads to the condition for solution

$$e^{\omega\delta t} = 1 + \frac{a\ \delta t}{\delta x^2}\left[e^{ik\delta x} + e^{-ik\delta x} - 2\right]$$

$$= 1 - \frac{2a\ \delta t}{\delta x^2}\,(1 - \cos k\ \delta x)$$

Thus $e^{\omega\delta t}$ is always ≤ 1; as long as it is also $\geq - 1$, the Fourier component will not grow in amplitude. To assure that this is the case for all values of k (the worst case is for $\cos k\ \delta x = - 1$) it is thus necessary that

$$\frac{a\ \delta t}{\delta x^2} < \frac{1}{2} \tag{7}$$

This is the well-known stability condition for the explicit heat diffusion difference approximation of Eq. (4); its validity has been proved in numerous numerical tests involving the actual sequential solving of (4).

The significance of this stability result can be seen as follows.

Essentially, it is a restriction on the size of δt, the time step interval. For a given set of temperatures at $t = n\delta t$, the difference $T_j^{n+1} - T_j^n$ depends linearly on δt. If, for example, the temperature profile at $t = n\delta t$ has a small fluctuation which is concave downwards, then the calculation will tend to decrease the temperature in the bump. For a small δt, the decrease in one cycle is small; for a larger δt, the bump could be exactly flattened out; for even larger δt, it could be reflected into a bump of similar shape and amplitude, but concave upwards. This last is the case of $e^{\omega\delta t} = -1$. For any larger δt the bump is reflected in one cycle into one of larger amplitude than before. This is the case of instability, since in each succeeding cycle, the bump alternately reflects back and forth with ever increasing amplitude.

In contrast, the difference approximation in (6) is stable for all values of δt. The analysis to prove this is the same as before, and leads to the condition for solution of a Fourier term

$$e^{\omega\delta t} = \frac{1}{1 + \frac{2a\delta t}{\delta x^2}(1 - \cos k\delta x)}$$

so that for any size of δt, $e^{\omega\delta t} \leq 1$. It would thus seem that one should always use the implicit form (6) or some other with similar stability properties, since then the solution could be advanced to the desired final time in a small number of cycles. There is, however, the additional consideration of accuracy. How well does the approximate solution represent the true solution of the differential equations?

In some cases, nothing is to be gained by the allowance of long time steps given by the unconditionally-stable forms, since the condition for accuracy is very close to that for the conditional stability.

One circumstance, however, in which the implicit form may be useful arises when the calculation is to be applied to a material with discontinuity in diffusion coefficient. Suppose, for example, that $a = a_L$ for $x_L \leq x < x_1$ and $a = a_R$ for $x_1 < x \leq x_R$. (The value that should be used at $x = x_1$ for the finite difference calculation need not be considered here. The matter is somewhat complicated and has been discussed elsewhere.[2]) Suppose further that $a_L \gg a_R$ so that the temperature diffuses much more rapidly in the left section than in the right section. If, then, heat enters the materials from the left, that side conducts it quickly to the boundary and may remain always at almost the same temperature. The right side will then have the large temperature gradients and contain all the features of interest. But to calculate the entire system with (4), one would have to be limited by the requirement $2a_L\delta t/\delta x^2 < 1$ which is much more stringent than the condition for the region of interest. Thus an implicit, unconditionally-stable procedure would be useful, with the size of δt determined by the requirements of accuracy on the right side, which would be a far less stringent restriction.

The difference approximation (5) might seem reasonable, on the argument that basing the new temperature in a cell on the average of adjacent old temperatures would have a smoothing effect and thus improve the stability. That this idea is wrong can be seen by repeating the analysis, which leads to the condition

$$e^{\omega \delta t} = \cos k\delta x - \frac{2a\delta t}{\delta x^2}(1 - \cos k\,\delta x)$$

which is unconditionally unstable, since there are values of k (for which $\cos k\delta x = -1$) leading to $e^{\omega\,\delta t} < -1$ for any non-zero intervals.

The form of the first term on the right of (5) is bad for another reason. The equation can be made unconditionally stable by changing the timing of the second term to n + 1, but the first term contributes an inaccuracy, which, for fixed δx, gets worse as $\delta t \to 0$! To see this, and to illustrate a method of accuracy analysis, consider (5) in the form it is written. Use the Taylor expansions, centered about j and n,

$$T_{j\pm 1}^{\,n} = T \pm \delta x \frac{\partial T}{\partial x} + \frac{\delta x^2}{2}\frac{\partial^2 T}{\partial x^2} \pm \frac{\delta x^3}{6}\frac{\partial^3 T}{\partial x^3} + \frac{\delta x^4}{24}\frac{\partial^4 T}{\partial x^4} + \cdots$$

$$T_j^{n+1} = T + \delta t \frac{\partial T}{\partial t} + \frac{\delta t^2}{2}\frac{\partial^2 T}{\partial t^2} + \cdots$$

Then (5) becomes, to order δt^2 and δx^2,

$$T + \delta t \frac{\partial T}{\partial t} + \frac{\delta t^2}{2}\frac{\partial^2 T}{\partial t^2} = T + \frac{\delta x^2}{2}\frac{\partial^2 T}{\partial x^2} + \frac{a\,\delta t}{\delta x^2}\left[\delta x^2 \frac{\partial^2 T}{\partial x^2} + \frac{\delta x^4}{12}\frac{\partial^4 T}{\partial x^4}\right]$$

or

$$\frac{\partial T}{\partial t} - a\frac{\partial^2 T}{\partial x^2} = \frac{\delta x^2}{2\delta t}\frac{\partial^2 T}{\partial x^2} + \frac{a\,\delta x^2}{12}\frac{\partial^4 T}{\partial x^4} - \frac{\delta t}{2}\frac{\partial^2 T}{\partial t^2} \tag{8}$$

The right side, which has the form of a source to the otherwise-conservative temperature field, expresses the lowest order errors. The last term can be rewritten, using the lowest order equation

$$\frac{\partial^2 T}{\partial t^2} = \frac{\partial}{\partial t}\left(a\,\frac{\partial^2 T}{\partial x^2}\right) = a\frac{\partial^2}{\partial x^2}\left(\frac{\partial T}{\partial t}\right) = a^2\,\frac{\partial^4 T}{\partial x^4}$$

Also we define

$$\epsilon \equiv \frac{a\,\delta t}{\delta x^2}$$

Then

$$\frac{\partial T}{\partial t} - a\,\frac{\partial^2 T}{\partial x^2} = \frac{a}{2\epsilon}\,\frac{\partial^2 T}{\partial x^2} + \frac{a\,\delta x^2}{2}\,\left(\frac{1}{6} - \epsilon\right)\,\frac{\partial^4 T}{\partial x^4} \tag{9}$$

The first term on the right side of (8) and (9) is particularly undesirable. With fixed δx, that error term <u>increases</u> as $\delta t \to 0$. Furthermore, for reasonable ϵ, it is larger than the real diffusion term and completely obscures the true solution. Such a term does not appear in the analogous expansion of (4) or (6).

Note that the term proportional to δt in (8) actually contributes one proportional to δx^2. Since δx^2 terms are already otherwise present, there is, therefore, no point in attempting removal of the δt term by a more careful time centering of the equation (such as in the implicit form

$$T_j^{n+1} - T_j^{n-1} = \frac{2\,\delta t}{\delta x^2}\left(T_{j+1}^{\;n} + T_{j-1}^{\;n} - 2\,T_j^n\right)$$

which actually has very bad stability properties — see next page —but from which terms of order δt cancel out in the Taylor expansion), unless such a procedure produces otherwise good results. Indeed the δt term would appear to be of value in increasing accuracy, since by the choice

$\epsilon = 1/6$, the δx^2 term can be made to vanish. Actually, however, the higher-order terms may be as important as the one which has been made to vanish. Usually the procedure of testing an approximation method is the most efficient way of determining its accuracy.

To conclude this example, we mention a comparison between two additional forms of the difference approximation to the diffusion equation. These are both "centered" in time and appear to resemble each other closely. These stability properties, however, are extremely different. They are

$$T_j^{n+1} = T_j^{n-1} + \frac{2a\delta t}{\delta x^2}\left[T_{j+1}^{\ n} + T_{j-1}^{\ n} - 2T_j^n\right]$$

$$T_j^{n+1} = T_j^{n-1} + \frac{2a\delta t}{\delta x^2}\left[\frac{1}{2}\left(T_{j+1}^{n+1} + T_{j-1}^{n+1} - 2T_j^{n+1}\right) + \frac{1}{2}\left(T_{j+1}^{n-1} + T_{j-1}^{n-1} - 2T_j^{n-1}\right)\right]$$

Application of the foregoing analysis shows that the first one is <u>unconditionally unstable</u>, while the second one is <u>unconditionally stable</u>.

<u>Example 2</u>. The wave equation in difference form exhibits somewhat different properties from those of the diffusion equation. We shall study

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{10}$$

in which u is the compression and c is the propogation speed. Define a function v by the equations

$$\frac{\partial v}{\partial t} + c\,\frac{\partial u}{\partial x} = 0 \tag{11}$$

$$\frac{\partial u}{\partial t} + c\,\frac{\partial v}{\partial x} = 0 \tag{12}$$

of which (10) is a consequence. The form (11), (12) is more convenient for differencing, and, together with additional terms, is a common form of equations of this type.

The differential stability properties can be determined from (10) or from (11) and (12) by study of a Fourier component of the difference between two solutions. We use (11) and (12) in analogy with the procedure for the difference approximation, and try the solution

$$u = u_o\, e^{ikx}\, e^{\omega t}$$

$$v = v_o\, e^{ikx}\, e^{\omega t}$$

Then,

$$\omega\, v_o + icku_o = 0$$

$$\omega\, u_o + ickv_o = 0$$

which can be solved for non-zero u_o and v_o only if $\omega = \pm\, ick$. Since ω is therefore purely imaginary, the difference between two solutions is composed entirely of oscillating components which neither damp nor amplify in time.

We shall consider two forms of the difference approximation to (11) and (12). The value of u_j for a cell will be an appropriate integral over the cell, similar to that used to define the temperature in Example 1.

$$u_j \delta x \equiv \int_{x_j - \frac{\delta x}{2}}^{x_j + \frac{\delta x}{2}} u \, dx$$

Thus (12) becomes

$$\frac{du_j}{dt} + \frac{c}{\delta x}\left[v(x_j + \frac{\delta x}{2}) - v(x - \frac{\delta x}{2})\right] = 0$$

Here $v(x_j + \frac{\delta x}{2})$ means the value of v at the point $x_j + \delta x/2$, as opposed to $v_{j+\frac{1}{2}}$ which we here define, in analogy to u_j, by

$$v_{j+\frac{1}{2}} \, \delta x \equiv \int_{x_j}^{x_j + \delta x} v dx$$

With this definition, we can similarly write for (11)

$$\frac{dv_{j+\frac{1}{2}}}{dt} + \frac{c}{\delta x}\left[u(x_j + \delta x) - u(x_j)\right] = 0$$

in which there is the analogous distinction between $u(x_j)$ which is the value of u at x_j, and u_j, the cell-wise average.

So far there is no approximation; the equations are exactly correct. We now introduce the approximate replacements

$$u(x_j) \rightarrow u_j$$

$$v(x_j + \delta x/2) \rightarrow v_{j+\frac{1}{2}}$$

whereby the equations become

$$\frac{du_j}{dt} + \frac{c}{\delta x}\left(v_{j+\frac{1}{2}} - v_{j-\frac{1}{2}}\right) = 0$$

$$\frac{dv_{j+\frac{1}{2}}}{dt} + \frac{c}{\delta x}\left(u_{j+1} - u_j\right) = 0$$

Finally with the same approximate representation of the time derivative as in Example 1, the equations become

$$\left.\begin{aligned} v_{j+\frac{1}{2}}^{n+1} &= v_{j+\frac{1}{2}}^{n} - \frac{c\,\delta t}{\delta x}\left(u_{j+1}^{n} - u_{j}^{n}\right) \\ u_{j}^{n+1} &= u_{j}^{n} - \frac{c\,\delta t}{\delta x}\left(v_{j+\frac{1}{2}}^{n} - v_{j-\frac{1}{2}}^{n}\right) \end{aligned}\right\} \qquad (13)$$

(Note: The reader who is interested in pursuing further the subject of deriving difference approximations to differential equations, may consult the papers by Taub[3] and Bromberg[4] for interesting reading.)

While the difference approximation (4) was conditionally stable, the analogous form (13) is unconditionally unstable. The reason for the difference in stability properties arising from essentially the same type of approximation comes from the difference between the respective differential stability properties. In the differential diffusion equation, perturbations are damped; while in the wave equation they remain at fixed amplitude. Thus the same type of difference approximation applied to both — in both cases causing a tendency towards amplification of perturbations — would bring the wave equation to immediate instability while the diffusion-equation stability would be protected by its possession of an initial degree of perturbation damping.

The instability of (13) is easily demonstrated by an examination of the trial solution

$$u_{j}^{n} = u_{o}\, e^{ikj\delta x}\, e^{n\omega\delta t}$$

$$v_{j+\frac{1}{2}}^{n} = v_{o}\, e^{ik(j+\frac{1}{2})\delta x}\, e^{n\omega\delta t}$$

Substitution into (13) leads to the pair of equations

$$v_o(e^{\omega\delta t} - 1) + u_o\left(\frac{2\ ic\delta t}{\delta x}\right)\sin\left(\frac{k\ \delta x}{2}\right) = 0$$

$$u_o(e^{\omega\delta t} - 1) + v_o\left(\frac{2\ ic\delta t}{\delta x}\right)\sin\left(\frac{k\ \delta x}{2}\right) = 0$$

for which the condition for non-trivial solution is

$$e^{\omega\delta t} = 1 \pm \frac{2\ ic\delta t}{\delta x}\sin\left(\frac{k\ \delta x}{2}\right)$$

This is a complex result. The stability properties are related to the magnitude of $e^{\omega\delta t}$, since the amplitude of u and v changes with time by the product of an oscillating factor with the quantity $|e^{\omega\delta t}|^n$.

Now

$$|e^{\omega\delta t}| = \sqrt{1 + \frac{4\ c^2\delta t^2}{\delta x^2}\ \sin^2\left(\frac{k\delta x}{2}\right)}$$

so that the magnitude will always be greater than unity for at least some Fourier components, no matter what non-zero values of δt and δx are used.

Conditional stability can be achieved through a simple modification of (13). In each time cycle, the new values of v are first computed, and these are used to find the new values of u,

$$\left.\begin{aligned} v_{j+\frac{1}{2}}^{n+1} &= v_{j+\frac{1}{2}}^{n} - \frac{c\ \delta t}{\delta x}\left(u_{j+1}^{n} - u_j^n\right) \\ u_j^{n+1} &= u_j^n - \frac{c\ \delta t}{\delta x}\left(v_{j+\frac{1}{2}}^{n+1} - v_{j-\frac{1}{2}}^{n+1}\right) \end{aligned}\right\} \qquad (14)$$

which lead to the stability analysis equation

$$\left(e^{\omega\delta t} - 1\right)^2 + e^{\omega\delta t}\left[\frac{4\ c^2\delta t^2}{\delta x^2}\sin^2\left(\frac{k\ \delta x}{2}\right)\right] = 0$$

Abbreviate the bracket factor by λ; then the solution for $e^{\omega \delta t}$ is

$$e^{\omega\delta t} = 1 - \tfrac{1}{2}\lambda \pm \tfrac{1}{2}\sqrt{\lambda^2 - 4\lambda}$$

For $\lambda < 4$, this is complex and has magnitude unity. For $\lambda > 4$ there are two magnitudes, one of which is always less than -1. Thus the difference equations are stable if $\lambda < 4$ for all values of k, that is, if

$$\left|\frac{c\ \delta t}{\delta x}\right| < 1^{*}$$

A further modification of (14) leads to unconditional stability,

$$\left.\begin{aligned} v^{n+1}_{j+\frac{1}{2}} &= v^{n}_{j+\frac{1}{2}} - \frac{c\ \delta t}{\delta x}\left(u^{n+1}_{j+1} - u^{n+1}_{j}\right) \\ u^{n+1}_{j} &= u^{n}_{j} - \frac{c\ \delta t}{\delta x}\left(v^{n+1}_{j+\frac{1}{2}} - v^{n+1}_{j-\frac{1}{2}}\right) \end{aligned}\right\} \qquad (15)$$

which equations are now implicit, and require special techniques for solution. With the same definition for λ, the stability analysis gives

$$e^{\omega\delta t} = \frac{1 \pm i\sqrt{\lambda}}{1 + \lambda}$$

from which

$$\left|e^{\omega\delta t}\right| = \frac{1}{\sqrt{1 + \lambda}}$$

which is always ≤ 1 since λ is always ≥ 0.

Example 3. The diffusing wave combines properties of the first two examples. The equations we use are an extension of (11) and (12)

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*This well-known restriction has been called the "Courant Condition."

$$\left.\begin{aligned}\frac{\partial v}{\partial t} + c\,\frac{\partial u}{\partial x} &= 0\\ \frac{\partial u}{\partial t} + c\,\frac{\partial v}{\partial x} &= a\,\frac{\partial^2 u}{\partial x^2}\end{aligned}\right\}\qquad(16)$$

and the difference approximation is similar to that in (13)

$$\left.\begin{aligned}v_{j+\frac{1}{2}}^{n+1} &= v_{j+\frac{1}{2}}^{n} - \frac{c\,\delta t}{\delta x}\left(u_{j+1}^{n} - u_{j}^{n}\right)\\ u_{j}^{n+1} &= u_{j}^{n} - \frac{c\,\delta t}{\delta x}\left(v_{j+\frac{1}{2}}^{n} - v_{j-\frac{1}{2}}^{n}\right) + \frac{a\,\delta t}{\delta x^2}\left(u_{j+1}^{n} + u_{j-1}^{n} - 2\,u_{j}^{n}\right)\end{aligned}\right\}(17)$$

The stability analysis leads to

$$e^{\omega\,\delta t} = 1 - \mu \pm \sqrt{\mu^2 - \lambda}$$

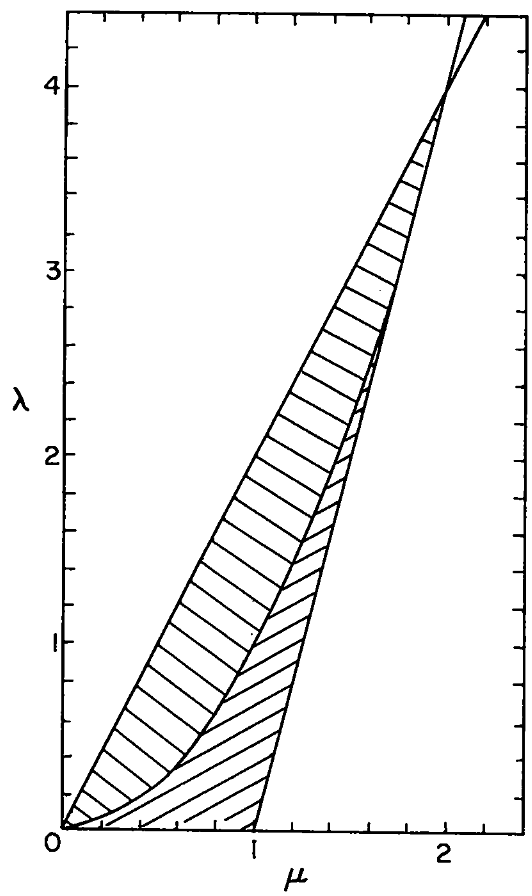
in which, as before,

$$\lambda \equiv \frac{4\,c^2\delta t^2}{\delta x^2}\,\sin^2\left(\frac{k\delta x}{2}\right)$$

and, in addition,

$$\mu \equiv \frac{2\,a\delta t}{\delta x^2}\quad\sin^2\left(\frac{k\delta x}{2}\right)$$

In the λ-μ plane, $e^{\omega\delta t}$ is complex but has magnitude ≤ 1 between the parabola $\lambda = \mu^2$ and the line $\lambda = 2\mu$. In the rest of the triangle, defined by the lines $\lambda = 2\mu$, $\lambda = 0$, and $\lambda = 4\mu - 4$, the value of $e^{\omega\,\delta t}$ is completely real, and lies between -1 and +1. Thus the



condition for stability is $2\lambda \leq 4\mu \leq 4 + \lambda$, or

$$\frac{2\, c^2 \delta t^2}{\delta x^2} \sin^2\left(\frac{k\, \delta x}{2}\right) \leq \frac{2a\delta t}{\delta x^2} \sin^2\left(\frac{k\, \delta x}{2}\right) \leq 1 + \frac{c^2 \delta t^2}{\delta x^2} \sin^2\left(\frac{k\, \delta x}{2}\right) \tag{18}$$

The left-hand inequality shows that the diffusion must be greater than a certain minimum in order that it overcome the instability effects of the wave equation. That inequality can be written simply

$$c^2 \delta t \leq a \tag{19}$$

The right-hand inequality is related to the conditional quality of the diffusion stability — see (7). The condition is, however, relieved somewhat by the wave motion; only for $c = 0$ does it become as stringent as (7). The worst cases are those for which $\sin^2 (k\delta x/2) = 1$, so that the stability conditions can be written

$$c^2 \delta t \leq a \leq \frac{1}{2}\left(\frac{\delta x^2}{\delta t} + c^2 \delta t\right)$$

PART II

STABILITY OF NON-LINEAR EQUATIONS

The examples in Part I have demonstrated some of the distinctive stability properties of linear difference equations. A perturbation of initial conditions produces a new solution which either diverges from or converges to the old solution (in either case exponentially) or else, on the borderline of stability, continues to differ from the old solution by a constant amount. If the equation is nonlinear, then the initial behavior for a very small perturbation may closely resemble that of a linear equation. If unstable, however, the nonlinear effects can change the rate of growth of amplitude, perhaps even prevent the disturbance from unbounded growth. On the other hand, nonlinearity can introduce at least one class of instability, not present in linear equations, in which the perturbation grows linearly in time, rather than exponentially.

We shall exhibit these features by means of two examples. No attempt will be made here to generalize the analysis, nor to place it on a rigorous basis. In both examples, validity of the analysis will be demonstrated by comparison with a numerical solution of the difference equations involved.

Example 1. Consider the coupled ordinary differential equations, in which y and z are functions of time t

$$\left.\begin{aligned} \frac{dy}{dt} &= 2z - \frac{\alpha}{2}\, y|y| \\ \frac{dz}{dt} &= -\, y \end{aligned}\right\} \qquad (20)$$

(This pair of equations was chosen because of its resemblance to the more complicated hydrodynamic equations in Example 2. These simpler equations exhibit parts of the features of interest and are examined first because of the relative ease of analysis.)

The nonlinear term containing α contributes the essential features of interest to the study. Without it the solutions would be perfectly periodic with arbitrary amplitude. Note that these equations can be combined to give

$$\frac{dE}{dt} = - \frac{\alpha}{2} y^2 |y| \tag{21}$$

where we have used the definition for E,

$$E \equiv \frac{1}{2} y^2 + z^2 \tag{22}$$

For any initial values of y and z, $E \to 0$ as $t \to \infty$, and so also do the amplitudes of y and z.

The time-difference approximation corresponding to (20) is

$$\left.\begin{aligned} y^{n+1} &= y^n + \delta t \left[2z^n - \frac{\alpha}{2} y^n |y^n| \right] \\ z^{n+1} &= z^n - y^n \delta t \end{aligned}\right\} \tag{23}$$

These equations can likewise be combined to show the change in E, as still defined in equation (22),

$$\frac{E^{n+1} - E^n}{\delta t} = - \frac{\alpha}{2} (y^n)^2 |y^n| + \delta t \left[\tfrac{1}{2}(2z^n - \frac{\alpha}{2} y^n |y^n|)^2 + (y^n)^2 \right] \tag{24}$$

Thus, there is a conflict between two terms, each of which always retains the same sign. The first term is cubic in the amplitude while the second, which is proportional to δt, contains quadratic through quartic parts. When the amplitude is small, the second term dominates, and E increases. As the amplitude thereby increases, the first term increases in magnitude and can eventually balance, on the average, the second term. For very large initial amplitude, the quartic part of the second term dominates; and the amplitude can then grow without bound.

The condition that $E^{n+1} = E^n$ is accomplished if

$$z^n = \frac{y^n}{2}\left[\frac{\alpha|y^n|}{2} \pm \sqrt{\frac{\alpha}{\delta t}|y^n| - 2}\ \right]$$

If $|y^n| < \frac{2\delta t}{\alpha}$, then there is no real value of z^n for which $E^{n+1} = E^n$; indeed, for any value of z^n, $E^{n+1} > E^n$ if $|y^n| < \frac{2\delta t}{\alpha}$. Thus the mean amplitude of y will asymptotically be at least $\frac{2\delta t}{\alpha}$.

To determine more precisely the equilibrium amplitude, we proceed as follows. First, determine the solution of equations (20) in the limit $\alpha \to 0$. An appropriate solution is

$$z = A \sin(t\sqrt{2}) \tag{25}$$

$$y = -A\sqrt{2}\cos(t\sqrt{2}) \tag{26}$$

We assume that the effects of finite α and δt do not change these solutions much, but only govern the equilibrium value of the amplitude A. Since we are thus working with small values of δt, we replace equations (23) and (24) by the lowest order approximations,

$$\frac{dy}{dt} + \frac{\delta t}{2}\frac{d^2y}{dt^2} = 2z - \frac{\alpha}{2}\,y|y| \tag{27}$$

$$\frac{dz}{dt} + \frac{\delta t}{2} \frac{d^2 z}{dt^2} = - y \tag{28}$$

$$\frac{dE}{dt} = - \frac{\alpha}{2} y^2 \, |y| - \frac{\delta t}{2} \left(y \frac{d^2 y}{dt^2} + 2z \frac{d^2 z}{dt^2} \right) \tag{29}$$

Putting the approximate solutions (25) and (26) into equation (29), we get

$$\frac{dE}{dt} = - \alpha A^3 \sqrt{2} \cos^2(t \sqrt{2}) |\cos(t \sqrt{2})| + 2 A^2 \delta t \tag{30}$$

At equilibrium, the average of dE/dt over a cycle of oscillation must vanish, so that

$$- \alpha A^3 \sqrt{2} \left(\frac{4}{3\pi}\right) + 2 A^2 \delta t = 0$$

or

$$A = A_\infty \equiv \frac{3\pi\delta t}{2\alpha\sqrt{2}} \tag{31}$$

Next we examine the manner in which A approaches the asymptotic value given by equation (31). With

$$\begin{aligned} E &= \tfrac{1}{2} y^2 + z^2 \\ &= A^2 \cos^2 (t \sqrt{2}) + A^2 \sin^2 t \sqrt{2}) \\ &= A^2 \end{aligned}$$

equation (30) becomes, averaged over a quarter cycle,

$$\frac{dA}{dt} = - \frac{2\alpha\sqrt{2}}{3\pi} A(A - A_\infty)$$

This has the solution

$$A = \frac{A_o A_\infty}{A_o + (A_\infty - A_o) e^{-t\delta t}} \tag{32}$$

where A_o is the value of A at t = 0. The time required to come significantly toward equilibrium is given by $t\delta t = 1$.

With this predicted behavior of A, we may proceed to derive a slightly more accurate solution for z and y. Putting equation (25) into equation (28), we get the final solution

$$\left.\begin{aligned} z &= A(t)\sin(t\sqrt{2}) \\ y &= -\sqrt{2}\,A(t)\cos(t\sqrt{2}) + A(t)\,\delta t\,\sin(t\sqrt{2}) \\ A(t) &= \frac{A_o A_\infty}{A_o + (A_\infty - A_o)\,e^{-t\delta t}} \\ A_\infty &= \frac{3\pi\delta t}{2\alpha\sqrt{2}} \end{aligned}\right\} \qquad (33)$$

and from this solution derive

$$E(t) = A^2 - \tfrac{1}{2}A^2\delta t\sqrt{2}\,\sin(2t\sqrt{2}) \qquad (34)$$

which exhibits the final amplitude and period of the oscillations of E.

Accuracy of the Solution

In order to test the results of the analytical approximations, a digital-computer code was written for solving equations (23) exactly. Computations were performed for various values of α and δt, as well as for a variety of initial values of y and z. In all cases examined, the behavior at late times was independent of the initial values, except as they affected the final phase of the oscillations. ("Late time" means after the execution of ten or more cycles of oscillation, corresponding to t values of 50 or more. In each case the calculation was run to t = 100 to assure equilibrium.) The amplitudes of y and z are shown as functions of $\delta t/\alpha$ in Fig. 1. For comparison, the theoretical solution is also shown; and the agreement is surprisingly good for values of $\delta t/\alpha$ up to 0.5. For any particular value of $\delta t/\alpha$, the agreement improves as δt decreases.

Part of a late-time cycle is shown in Fig. 2. The phase difference between y and z is not quite $\pi/2$, in agreement with the prediction of equations (33). Indeed, from those equations, y can be written

$$y = - A \sqrt{2 + \delta t^2} \quad \cos\left[t \sqrt{2} + \tan^{-1} \frac{\delta t}{\sqrt{2}}\right]$$

so that the minimum of y should precede the zero of z by a difference in t of approximately $\delta t/2$. For the case in Fig. 2, with $\delta t = 0.2$, the displacement should be 0.1, while an average of 0.12 is actually observed.

A more crucial test of the theory is given by a comparison of the time variation of E. Fig. 3 illustrates this for one of the calculations. The phase difference between theoretical and observed oscillations was so small that it could not be well shown on the figure. The envelope of the theoretical E oscillations, wherein the discrepancy is more appreciable, is shown instead. For late times the mean theoretical and observed E values agreed to within much less than 1%, but the amplitude of observed oscillations exceeded the theoretically predicted amplitude by about the same amount as shown for the latest times shown in Fig. 3.

We remark in passing that there are at least two simple modifications which greatly improve the stability of the difference equations while still retaining the advantages of explicit form. They bear direct analogy to the changes shown in equation (14) of Part I.

$$\left.\begin{aligned} y^{n+1} &= y^n + \delta t\left[2z^{n+\mu} - \frac{\alpha}{2}\, y^n|y^n|\right] \\ z^{n+1} &= z^n - \delta t\; y^{n+\nu} \end{aligned}\right\} \qquad (35)$$

where, either $\mu = 1$, $\nu = 0$, and the second equation is solved first; or $\mu = 0$, $\nu = 1$, and the first equation is solved first. In a test of the second case, for example, the late-time amplitude of oscillation was cut by more than a factor of 10 over the amplitude with $\mu = \nu = 0$.

Proof of greater stability can be accomplished by an analysis similar to that used in deriving the amplitude of oscillations. Analogous to equation (18), the result is

$$A_\infty = \frac{3\pi\delta t}{2\alpha\sqrt{2}}\,(1 - \mu - \nu)$$

<u>Example 2</u>. We work with one-dimensional hydrodynamic equations in the form

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = -\,c^2\,\frac{\partial \varphi}{\partial x} + \nu\,\frac{\partial^2 u}{\partial x^2} \qquad (36)$$

$$\frac{\partial \varphi}{\partial t} + u\,\frac{\partial \varphi}{\partial x} + \frac{\partial u}{\partial x} = 0 \qquad (37)$$

where

$$\varphi \equiv \ln\left(\frac{\rho}{\rho_o}\right)$$

ν is an "artificial viscosity" coefficient (here a function of velocity only),[5] u and ρ are respectively velocity and density, and c is sound speed. (These equations are not quite correct since the term $\frac{1}{\rho}\frac{\partial\nu\rho}{\partial x}\frac{\partial u}{\partial x}$ has been neglected, thus simplifying the analysis but still leaving the qualitative features we wish to exhibit.) In example 3 of Part I, the linearized form of these equations was studied.

The spatial domain of the problems is divided into cells numbered $j = 1,2,\cdots,J$. The difference equations, to first order in δt, analogous to (27) and (28), are

$$\frac{du_j}{dt} = -\frac{\delta t}{2}\frac{d^2u_j}{dt^2} - \frac{u_j}{2\delta x}(u_{j+1} - u_{j-1}) - \frac{c^2}{2\delta x}(\varphi_{j+1} - \varphi_{j-1}) + \frac{\nu_j}{\delta x^2}(u_{j+1} + u_{j-1} - 2u_j) \tag{38}$$

$$\frac{d\varphi_j}{dt} = -\frac{\delta t}{2}\frac{d^2\varphi_j}{dt^2} - \frac{u_j}{2\delta x}(\varphi_{j+1} - \varphi_{j-1}) - \frac{1}{2\delta x}(u_{j+1} - u_{j-1}) \tag{39}$$

To determine the equilibrium amplitudes of oscillation, we follow the same procedure as in Example 1. This will provide some useful results, but further on we shall demonstrate a fallacy in the argument and show how the error is to be corrected. Define

$$E \equiv \sum_{j=1}^{J} E_j \equiv \frac{1}{2}\sum_{j=1}^{J}(u_j^2 + c^2\varphi_j^2)$$

Then

$$\frac{dE_j}{dt} = -\frac{\delta t}{2}\left(u_j\frac{d^2u_j}{dt^2} + c^2\varphi_j\frac{d^2\varphi_j}{dt^2}\right) - \frac{u_j^2}{2\delta x}(u_{j+1} - u_{j-1}) - \frac{c^2u_j\varphi_j}{2\delta x}(\varphi_{j+1} - \varphi_{j-1}) - \frac{c^2}{2\delta x}[u_j(\varphi_{j+1} - \varphi_{j-1}) + \varphi_j(u_{j+1} - u_{j-1})] + \frac{\nu_ju_j}{\delta x^2}(u_{j+1} + u_{j-1} - 2u_j) \tag{40}$$

Now a solution of the linearized form of equations (38) and (39), in the limit as $\delta t \to 0$, and with $\nu = 0$, is

$$\left.\begin{aligned} \varphi_j &= \varphi_o \cos \omega t \cos j\sigma \\ u_j &= c\varphi_o \sin \omega t \sin j\sigma \end{aligned}\right\} \tag{41}$$

where

$$\left.\begin{aligned} \sigma &\equiv k\ \delta x \\ \omega &\equiv \frac{c}{\delta x} \sin \sigma \end{aligned}\right\} \tag{42}$$

and φ_o is an arbitrary amplitude. To be specific we pick boundary conditions such that the ends of the spatial domain are at rest; $u_1 \equiv u_J \equiv 0$, so that $\sigma = \frac{\zeta\pi}{J}$, $\zeta = 1,2, \cdots, J-1$. This solution is put into (40) and averaged over a cycle of oscillation. Many of the terms vanish. The result is also to be summed over the cells of the system and leads to the equation, in which φ_o must now be considered to vary with time,

$$\varphi_o \frac{d\varphi_o}{dt} = \frac{\omega^2 \delta t}{2} \varphi_o^2 + \frac{2}{Jc^2 \delta x^2} \sum_{j=1}^{J} < \nu_j u_j (u_{j+1} + u_{j-1} - 2\, u_j) > \tag{43}$$

in which $< >$ signifies time average.

Consider first the case $\nu \equiv$ constant. Then the sum becomes

$$\nu \sum_{j=1}^{J} < u_j(u_{j+1} + u_{j-1} - 2u_j) >$$

$$= c^2 \nu\, \varphi_o^2 \sum_{j=1}^{J} \sin j\sigma\, [\sin(j+1)\,\sigma + \sin(j-1)\,\sigma - 2 \sin j\sigma] < \sin^2 \omega t >$$

$$= c^2 \nu\, \varphi_o^2 \sum_{j=1}^{J} \sin^2 j\sigma(\cos \sigma - 1) = - \frac{J}{2} c^2 \nu\, \varphi_o^2 (1 - \cos \sigma)$$

where terms of order unity have been neglected as small compared with terms of order J. Thus (43) becomes

$$\frac{d\varphi_o}{dt} = \frac{\varphi_o \nu \sin^2\sigma}{2\delta x^2}\left[\frac{c^2\delta t}{\nu} - \frac{2(1 - \cos\sigma)}{\sin^2\sigma}\right]$$

This leads to the stability requirement

$$\frac{c^2\delta t}{\nu} \leq \frac{2(1 - \cos\sigma)}{\sin^2\sigma}$$

which is the analogy of (19), and is precisely the same result that would be obtained by application of the method of Part I to the linearized equations. (The difference between this result and (19) arises from the difference in space differencing in the two cases.)

Consider now the results also for the Particle-in-cell type viscosity[5]

$$\nu = \frac{\alpha}{2}\,\delta x \left|u_j\right|$$

The appropriate averages and sums can be performed in the same manner as before, leading to the result

$$\frac{d\varphi_o}{dt} = -\,K\,\varphi_o(\varphi_o - \bar{\varphi}_o)$$

where

$$\left.\begin{aligned} K &\equiv \left(\frac{2\alpha c}{\delta x}\right)\left(\frac{4}{3\pi}\right)^2 (1 - \cos\sigma) \\ \bar{\varphi}_o &\equiv \left(\frac{c\delta t}{\alpha\delta x}\right)\left(\frac{3\pi}{4}\right)^2 \frac{(1 - \cos 2\sigma)}{8(1 - \cos\sigma)} \end{aligned}\right\} \qquad (44)$$

The equation can be integrated and it is found that the time required for the amplitude to grow from an initial value of φ_{oo} to half of the asymptotic value is

$$t = \frac{4\ \delta x^2}{c^2 \delta t(1 - \cos^2\sigma)} \quad \ln \frac{\bar{\varphi}_o - \varphi_{oo}}{\varphi_{oo}}$$

To test these conclusions, a program was written for high-speed computer to solve the difference equations through sufficient successive cycles to obtain equilibrium. The initial disturbance had a wave length of four cells, $\sigma = \pi/2$. The results showed good agreement in most respects with those predicted above. The one serious discrepancy occurred in the value of φ in every other cell. In those special cells, the value of φ increased without bound at nearly a constant rate. This instability had no effect on the rest of the cells, whose equilibrium fluctuation amplitudes were nearly at the predicted values. [They were lower than predicted by about 15% for the two values of $c\delta t/\alpha\delta x = 0.5$ and 0.2. The discrepancy is probably due to the neglect of phasing differences in the trial solution (41), which diagnosis is indicated by the form of the improved trial solution which follows - see (45)].

To explain the instability which the above analysis fails to reveal, it is necessary to re-examine the trial solutions, and to find a more accurate set than those given in (41). Since the machine calculation showed good agreement with the predicted u function in (41), it is reasonable to retain that function and look for a more accurate φ function. To do this, we substitute

$$u_j = c\varphi_o \sin \omega t \sin j\sigma$$

$$\varphi_j = \varphi_o \cos \omega t \cos j\sigma + q_j$$

into (39), and look for a solution for q_j. The resulting equation to be solved is

$$\frac{dq_j}{dt} + \frac{\delta t}{2}\frac{d^2 q_j}{dt^2} - \frac{\omega^2 \varphi_o \delta t}{2} \cos \omega t \cos j\sigma$$

$$= - \frac{c\varphi_o}{2\delta x} (q_{j+1} - q_{j-1}) \sin \omega t \sin j\sigma$$

$$+ \frac{c\varphi_o^2}{\delta x} \sin \omega t \cos \omega t \sin^2 j\sigma \sin \sigma$$

We assume that an expansion can be made in powers of φ_o

$$q_j = \varphi_o\, r \cos j\sigma + \varphi_o^2\, p \sin^2 j\sigma \sin \sigma$$

where p and r are functions of time only, and are determined by solving the equations

$$\frac{dr}{dt} + \frac{\delta t}{2}\frac{d^2 r}{dt^2} = \frac{\omega^2 \delta t}{2} \cos \omega t$$

$$\frac{dp}{dt} + \frac{\delta t}{2}\frac{d^2 p}{dt^2} = \frac{c}{\delta x} \sin \omega t\, (r + \cos \omega t)$$

Thus dr/dt is of order δt, as is also d^2r/dt^2. For consistency, we must here drop terms of order δt^2; the resulting equation has the solution

$$r = \frac{\omega \delta t}{2} \sin \omega t$$

It is the resulting $\sin^2\omega t$ term in the p equation which contributes to the instability we are seeking. Discarding the oscillating terms, we keep only that part which leads to the solution

$$p = \frac{c\omega\,\delta t}{4\delta x}\,t$$

so that, finally (with $\omega \equiv c \sin \sigma/\delta x$),

$$\begin{aligned}\varphi_j &= \varphi_o \cos j\sigma[\cos \omega t + \frac{\omega\,\delta t}{2} \sin \omega t] \\ &\quad + \varphi_o^2\left(\frac{c^2\delta t}{4\delta x^2}\right) t \sin^2 j\sigma \sin^2\sigma\end{aligned} \tag{45}$$

plus oscillating terms of order φ_o^2.

This solution has just the behavior necessary to explain the anomalous growth of φ in every other cell which occurred in the test with $\sigma = \pi/2$. For the cell with $j = 1$, for example, this result predicts

$$\varphi_1 = \frac{\varphi_o^2\ c^2\delta t}{4\ \delta x^2}\,t \tag{46}$$

This consistent growth of φ arises from a persistent coupling in the nonlinear term $u\ \partial\varphi/\partial x$, which became the second term on the right side of (39). Had (39) been in conservative form, this difficulty would not have arisen.

In the test problems we used $c = 1$, $\delta x = 1$. For $\delta t = 0.5$, the machine calculations gave $\varphi_o = 0.60$ (theoretical $\varphi_o = 0.69$); while for $\delta t = 0.2$ the machine gave $\varphi_o = 0.235$ (theoretical $\varphi_o = 0.278$). With the observed φ_o values, $d\varphi_1/dt$ was predicted to be, respectively, 0.045

and 0.0028. Since the initial behavior of φ_1 depends upon initial conditions of the problem, it should be expected that this slope would be reached only after an initial adjustment. Comparisons between the above analysis and the machine calculations are shown in Figs. 4 and 5. Scatter of the machine-calculated points is not followed by the results of analysis because of neglect of oscillating terms in the latter. In both cases, agreement of the slope prediction is excellent.

A modified procedure, analogous to that in (14), was tried on the computing machine to see if the amplitude of oscillations could be decreased. In each cycle, the new values of velocity were computed first, and these new values were used in finding the new values of φ. The result was a cut in amplitude by more than a factor of ten, and while the every-other-cell instability in φ was still present, its rate of growth was greatly reduced.

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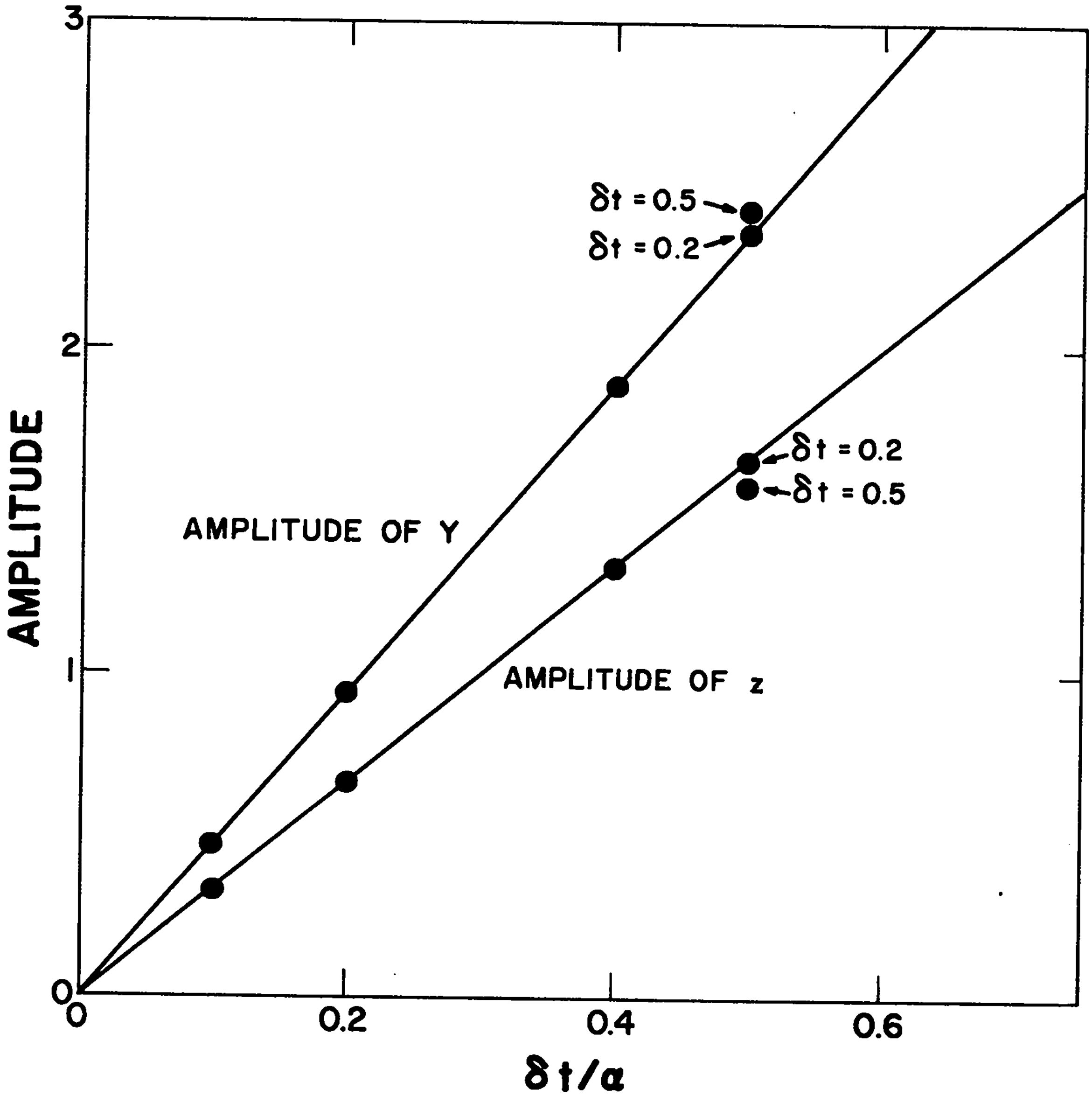

Fig. 1: Theoretical (straight line) and observed(points) amplitudes of y and z for various values of $\delta t/\alpha$. In all cases $\alpha = 1$ except that for which $\delta t = 0.2$ is explicitly marked.

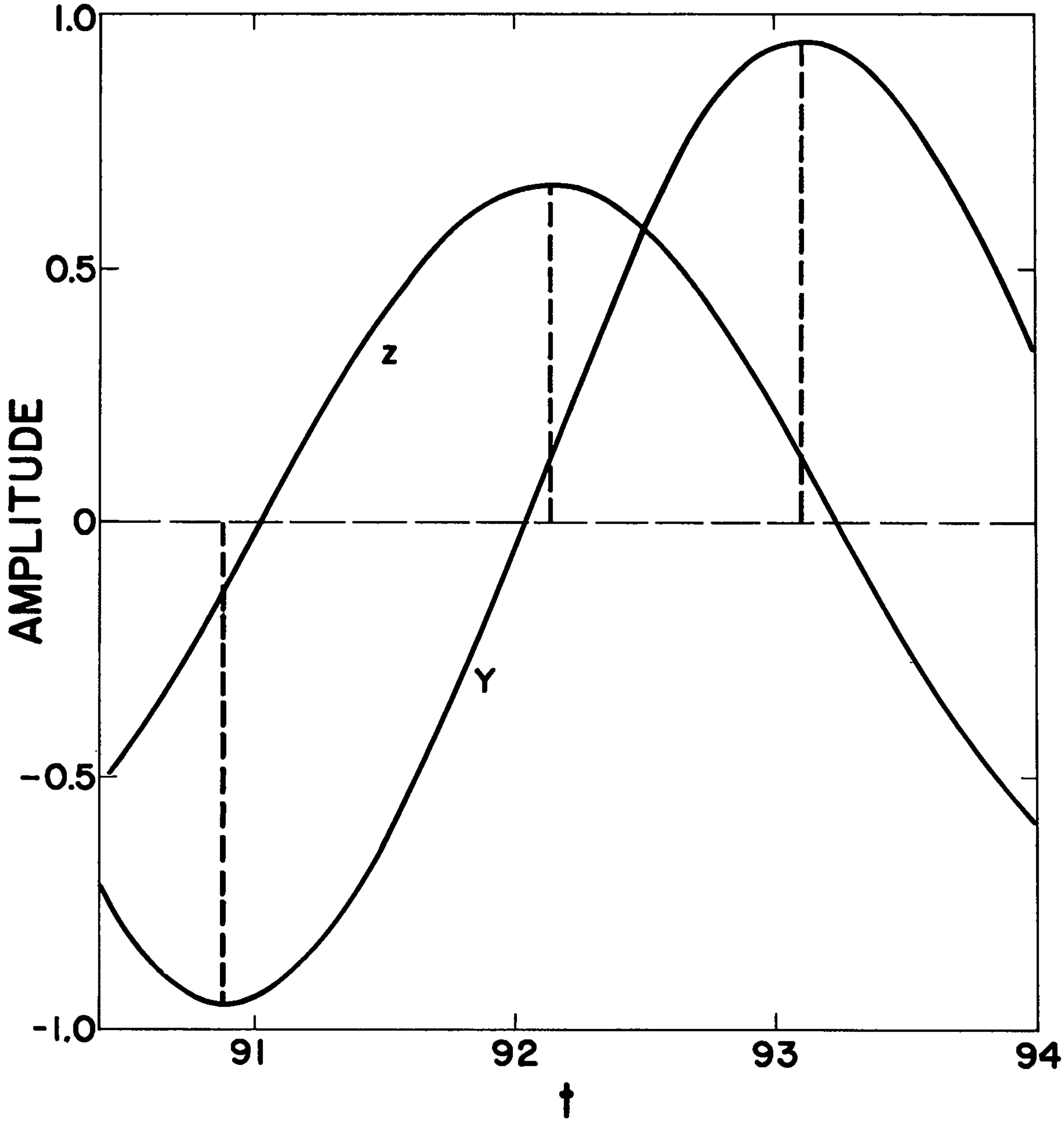

Fig. 2: Part of a late-time cycle of y and z showing their amplitude and phase relationships.

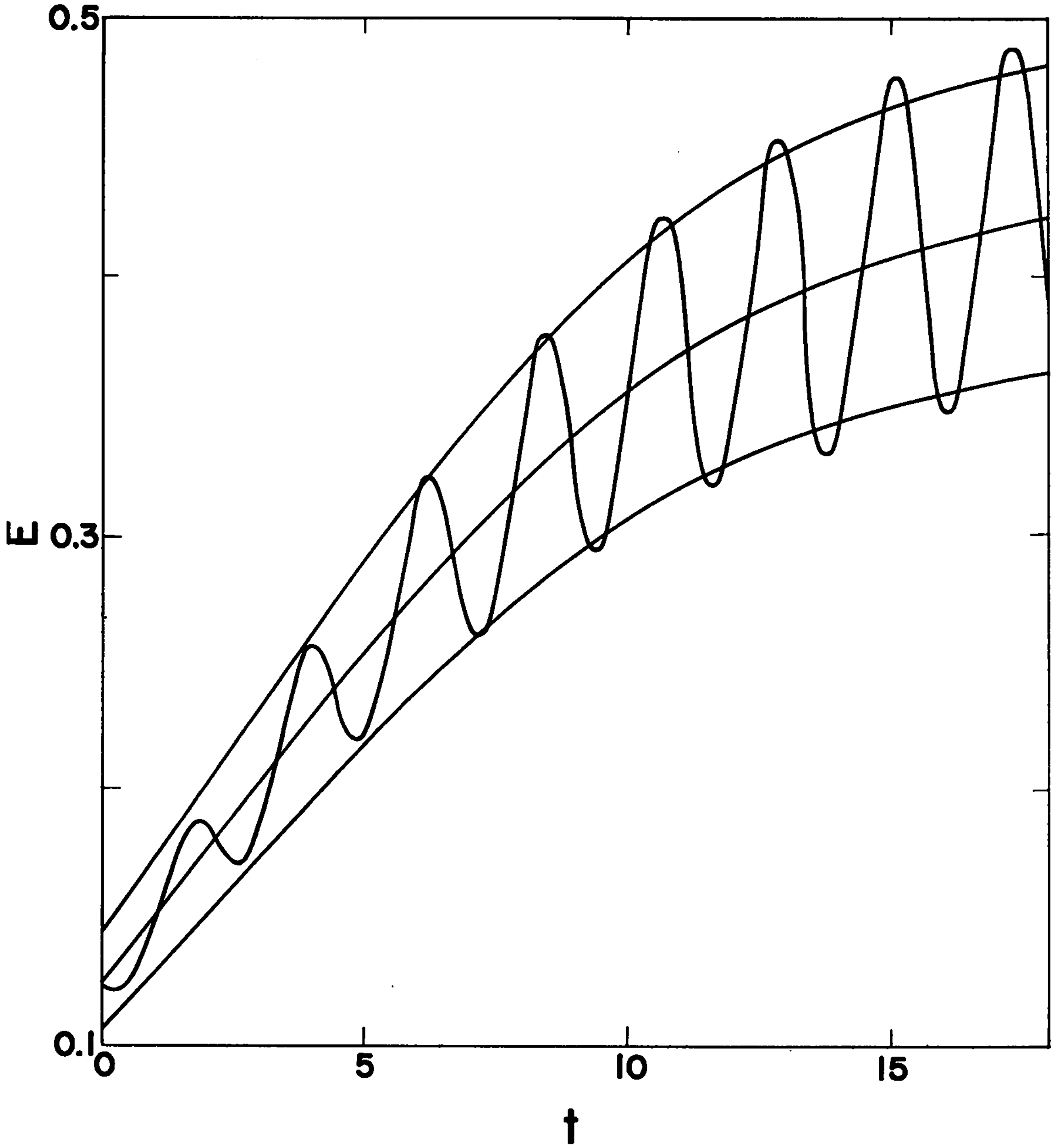

Fig. 3: Early-time variation of E as observed (oscillating line) and as predicted (shown by envelope of oscillations and mean thereof).

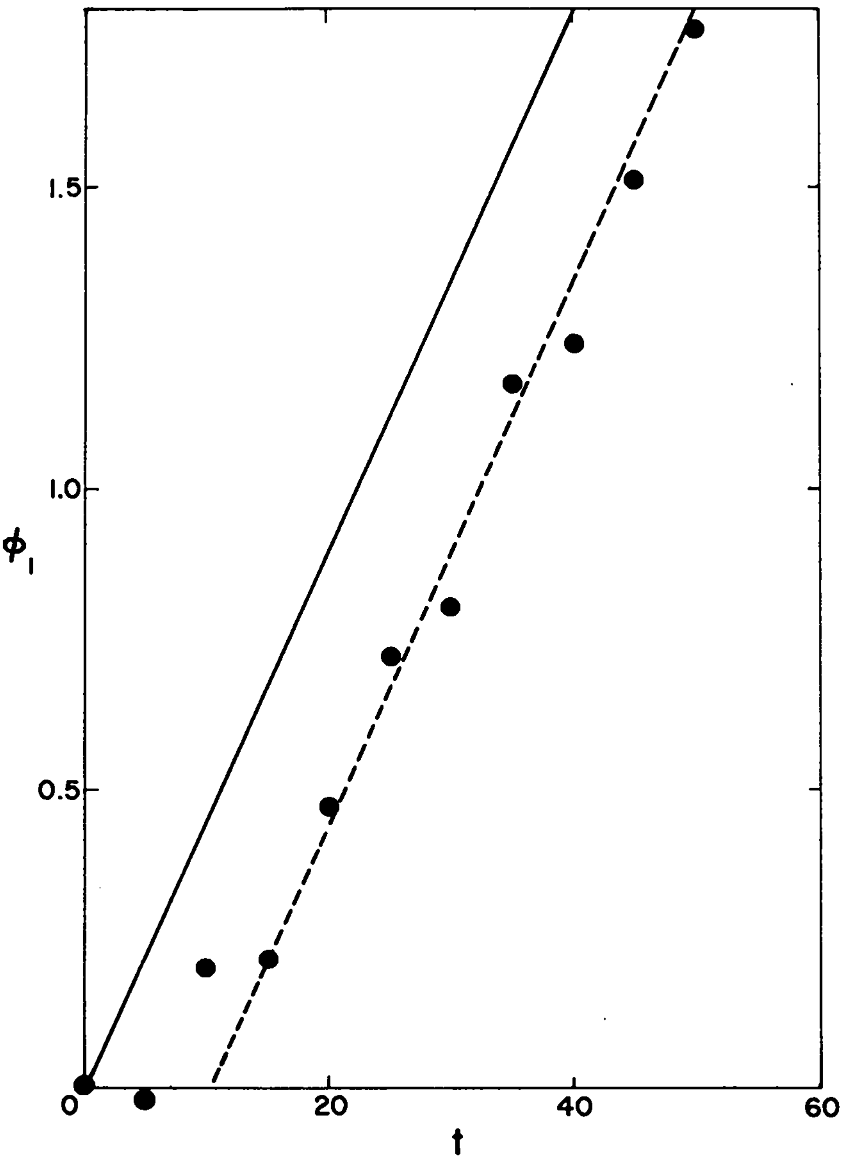

Fig. 4: The value of φ_1 as a function of time for $\delta t = 0.5$. Points are from machine calculation, solid line is from Eq. (46), dashed line is that result transposed.

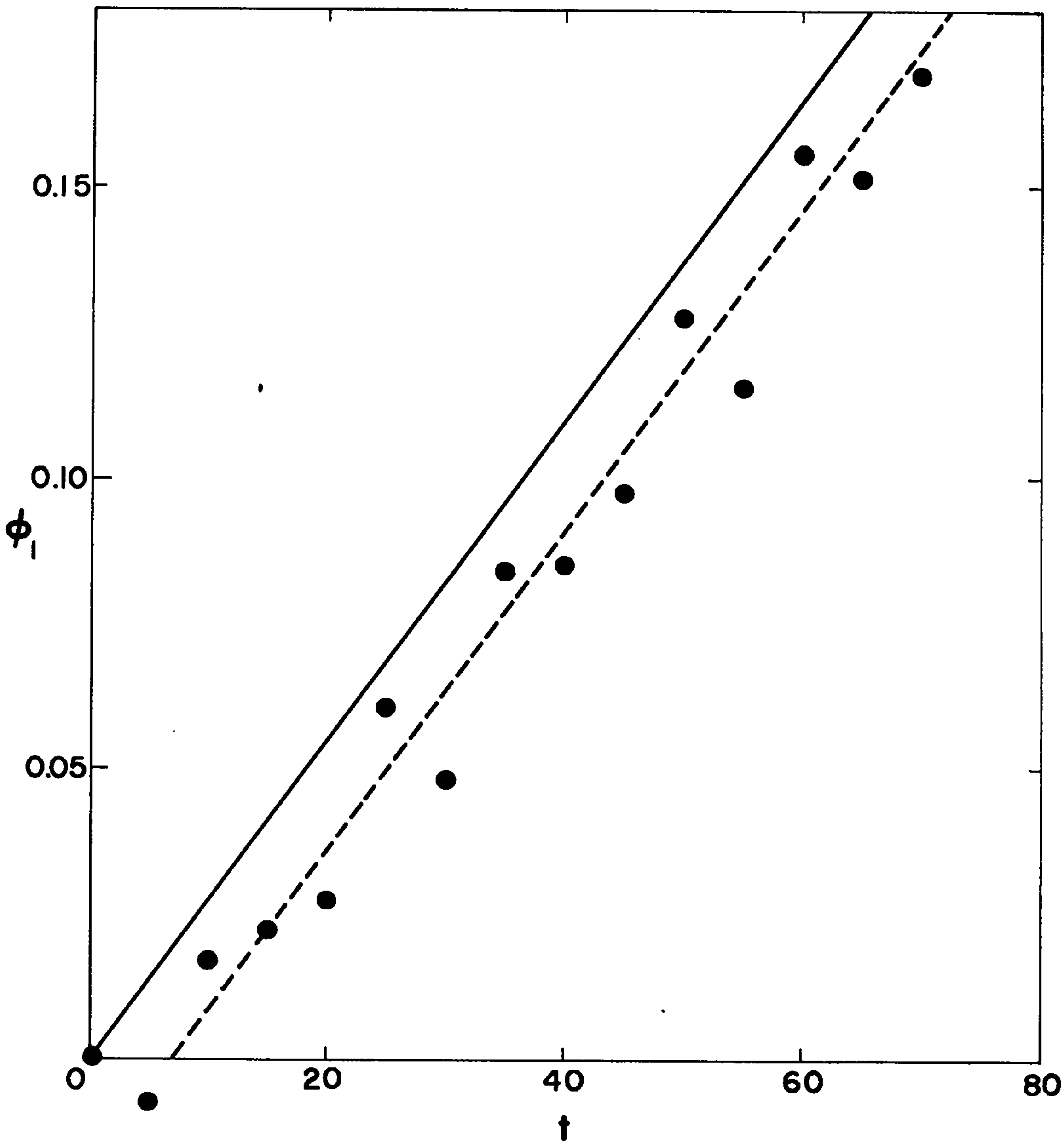

Fig. 5: The value of φ_1 as a function of time for $\delta t = 0.2$. Points are from machine calculation, solid line is from Eq. (46), dashed line is that result transposed.